J. Appl. Prob. **29**, 996–1002 (1992)
Printed in Israel

ASYMPTOTIC VARIANCE PARAMETERS FOR THE BOUNDARY LOCAL TIMES OF REFLECTED BROWNIAN MOTION ON A COMPACT INTERVAL

R. J. WILLIAMS,* *University of California at San Diego, La Jolla*

Abstract

A direct derivation is given of a formula for the normalized asymptotic variance parameters of the boundary local times of reflected Brownian motion (with drift) on a compact interval. This formula was previously obtained by Berger and Whitt using an $M/M/1/C$ queue approximation to the reflected Brownian motion. The bivariate Laplace transform of the hitting time of a level and the boundary local time up to that hitting time, for a one-dimensional reflected Brownian motion with drift, is obtained as part of the derivation.

LOCAL TIME; BOUNDARY REGULATOR PROCESS; LAPLACE TRANSFORM; MARTINGALE

AMS 1991 SUBJECT CLASSIFICATION: PRIMARY 60J65
SECONDARY 60H05, 60J55, 60K25

In [2], Berger and Whitt studied diffusion approximations to a rate-control throttle and the $G/G/1/C$ queue. These diffusions are one-dimensional reflected Brownian motions (with drift) on the interval $[0, C]$, where $C < \infty$. Amongst the formulas derived in [2] were expressions for the normalized asymptotic variance parameters of the boundary local times (or boundary regulator processes) of these reflected Brownian motions (see (4.14) of [2] and (1)–(6) below). These normalized asymptotic variance parameters are of interest because they are the squared coefficients of variation (SCVs, variance divided by the square of the mean) for the constituent i.i.d. random variables in a renewal process approximation to the overflow processes in the rate-control throttle and queueing models considered by Berger and Whitt [2]. Berger and Whitt obtained the expressions for these asymptotic variance parameters by first showing that they are limits of quantities for $M/M/1/C$ queues and then explicitly computing the $M/M/1/C$ quantities ([2], Theorem 3.1) and their limits ([2], (4.36) of Theorem 4.5). Ward Whitt asked the author whether one could not perhaps calculate the expressions for the

Received 8 August 1991; revision received 20 September 1991.
* Postal address: Department of Mathematics, University of California, San Diego, La Jolla, CA 92093-0112, USA.
Research supported in part by NSF Grants DMS 8657483 and 8722351, an Alfred P. Sloan Research Fellowship, and grants from AT&T Bell Labs and SUN Microsystems.

parameters in (4.14) of [2] directly from an analysis of the reflected Brownian motion. In this note it is shown that this is indeed possible. The objective is to compute a quantity (Γ defined in (5) below) which is made up of moments and joint moments of the hitting time of a level and the boundary local time up to that hitting time for a one-dimensional reflected Brownian motion with drift. The moments and joint moments are computed by differentiation of a bivariate Laplace transform. A formula for the latter is obtained by a deft choice of a suitable martingale. The symbolic differentiation of the Laplace transform was performed using the computer software package Mathematica on a SUN 3/60.

We begin by defining the reflected Brownian motion (RBM) under consideration. Fix $C > 0$. Let X be a one-dimensional Brownian motion with drift $\mu \in \mathbb{R}$, variance parameter $\sigma^2 > 0$, and suppose $X(0) = 0$. Define $\mathscr{F}_t = \sigma\{X(s): 0 \leqq s \leqq t\}$ for all $t \geqq 0$. A process will be called adapted if it is adapted to $\{\mathscr{F}_t\}$. There is a unique pair of non-decreasing, continuous, adapted, one-dimensional processes (L, U) such that

(i) $Z(t) \equiv X(t) + L(t) - U(t) \in [0, C]$ for all $t \geqq 0$,

(ii) L can increase only when Z is at 0,

(iii) U can increase only when Z is at C.

Existence and uniqueness follow easily from the more general work of Lions and Sznitman [6], or a 'bare-hands' proof can be given by successive application of the one-sided reflection mapping ([3], Chapter 8) using a sequence of stopping times. The process Z is referred to as a (μ, σ^2) reflected Brownian motion on $[0, C]$ with starting point at the origin. The processes L and U are called the local times of Z at the origin and C respectively. Now, define $\tau = \inf\{t \geqq 0 : Z(t) = C\}$, $\tau' = \inf\{t \geqq \tau : Z(t) = 0\} - \tau$. As claimed in ([2], §4.2), by a regenerative analysis and the central limit theorem for renewal processes, one has

$$a \equiv \lim_{t\to\infty} \frac{L(t)}{t} = \frac{E[L(\tau+\tau')]}{E[\tau+\tau']} \tag{1}$$

$$b \equiv \lim_{t\to\infty} \frac{U(t)}{t} = \frac{E[U(\tau+\tau')]}{E[\tau+\tau']} \tag{2}$$

$$\lim_{t\to\infty} \frac{\mathrm{Var}(L(t))}{t} = \sigma_L^2 \quad \text{and} \quad \lim_{t\to\infty} \frac{\mathrm{Var}(U(t))}{t} = \sigma_U^2 \tag{3}$$

where σ_L^2, σ_U^2 are called the asymptotic variance parameters of L and U respectively, and the associated normalized asymptotic variance parameters Γ_L, Γ_U defined by

$$\Gamma_L = a^{-1}\sigma_L^2 \quad \text{and} \quad \Gamma_U = b^{-1}\sigma_U^2 \tag{4}$$

are the squared coefficients of variation for the i.i.d. random variables constituting renewal process approximations to the overflow processes in the models considered by Berger and Whitt ([2], (4.13)). In fact, it turns out that $\Gamma_L = \Gamma_U$ (see [2], (4.14) or Remark 2 following Theorem 2 below), so we shall henceforth only consider Γ_L and denote it simply by Γ. By a regenerative analysis ([2], (4.45)) and the fact that L does not increase on $[\tau, \tau + \tau')$,

$$\Gamma = E\left[\left(L(\tau) - \frac{(\tau+\tau')E[L(\tau)]}{E[\tau+\tau']}\right)^2\right] \Big/ E[L(\tau)]. \tag{5}$$

The main object of this note is to verify formula (4.14) of [2], namely,

$$\Gamma = \begin{cases} 2C/3, & \text{if } \mu = 0 \\ \dfrac{2(1-\exp(2\theta C)) + 4\theta C\exp(\theta C)}{-\theta(1-\exp(\theta C))^2}, & \text{if } \mu \neq 0, \end{cases} \tag{6}$$

where $\theta = 2\mu/\sigma^2$. In order to do this, the first and second moments of τ, τ' and $L(\tau)$ need to be computed, as well as the first moment of $\tau L(\tau)$.

Note that if $\sigma > 0$ and $\tilde{Z}$ is a $(\mu/\sigma, 1)$ reflected Brownian motion on $[0, C/\sigma]$, then by Brownian scaling and the representation of the form (i) for $\tilde{Z}$, we have $Z \equiv \sigma\tilde{Z}$ is a (μ, σ^2) reflected Brownian motion on $[0, C]$ and the local time L of Z at the origin is equal to $\sigma\tilde{L}$ where $\tilde{L}$ is the local time of $\tilde{Z}$ at the origin. It follows that it suffices to verify (6) for $\sigma^2 = 1$. Thus we henceforth assume $\sigma^2 = 1$.

The following theorem is the key to our computation of Γ. The constituent moments required to compute Γ, that can be derived from (7), are displayed in Theorem 2, towards the end of this paper. These moments also enable one to compute all of the asymptotic quantities defined in (1)–(4) above.

Theorem 1. *For each* $\alpha, \beta \geqq 0$ *we have*

$$E[\exp(-\alpha L(\tau) - \beta\tau)] = \begin{cases} \dfrac{1}{1+\alpha C} & \text{if } \mu = 0, \beta = 0 \\ \dfrac{1}{\cosh(\gamma C) + \dfrac{\alpha}{\gamma}\sinh(\gamma C)} & \text{if } \mu = 0, \beta \neq 0 \\ \dfrac{e^{\mu C}}{\cosh(\gamma C) + \dfrac{(\alpha+\mu)}{\gamma}\sinh(\gamma C)} & \text{if } \mu \neq 0, \end{cases} \tag{7}$$

where $\gamma = (\mu^2 + 2\beta)^{1/2}$.

Proof. Consider the function $f: \mathbb{R} \times \mathbb{R}_+^2 \to \mathbb{R}$ defined by

$$\begin{aligned} &f(x, l, t) \\ &\quad = \exp(-\mu(x+l))[\gamma\cosh(\gamma(x+l)) + (\alpha+\mu)\sinh(\gamma(x+l))]\exp(-\alpha l - \beta t). \end{aligned} \tag{8}$$

It can be readily verified that f satisfies:

$$\tfrac{1}{2} f_{xx} + \mu f_x + f_t = 0 \tag{9}$$

and

$$f_l(x, -x, t) = 0 \qquad \text{for all } (x, t) \in \mathbb{R} \times \mathbb{R}_+. \tag{10}$$

Then by Itô's formula we have for $B_s \equiv X_s - \mu s$,

$$\begin{aligned} & f(X_{t\wedge\tau}, L_{t\wedge\tau}, t\wedge\tau) - f(0,0,0) \\ &= \int_0^{t\wedge\tau} f_x(X_s, L_s, s)dB_s + \int_0^{t\wedge\tau} f_l(X_s, L_s, s)dL_s \\ &\quad + \int_0^{t\wedge\tau} (\tfrac{1}{2} f_{xx} + \mu f_x + f_s)(X_s, L_s, s)ds. \end{aligned} \tag{11}$$

Since U does not increase prior to the time τ and $L(\cdot \wedge \tau)$ can increase only when $Z(\cdot \wedge \tau) = X(\cdot \wedge \tau) + L(\cdot \wedge \tau)$ is at the origin, it follows that the second integrand in (11) can be replaced by $f_l(X_s, -X_s, s)$, and then using (9)–(10) we see that

$$f(X_{t\wedge\tau}, L_{t\wedge\tau}, t\wedge\tau) = f(0,0,0) + \int_0^{t\wedge\tau} f_x(X_s, L_s, s)dB_s, \tag{12}$$

which is a local martingale. In fact, since $Z_s = X_s + L_s \in [0, C]$ for $0 \leqq s \leqq \tau$ and $\alpha L_s + \beta s \geqq 0$, it follows from (8) that the left member of (12) is bounded by a constant independent of t, and hence that it is actually a martingale (cf. Proposition 1.8 of [3]). Indeed, since $\tau < \infty$ a.s., after taking expectations in (12) and letting $t \to \infty$ we obtain by bounded convergence that $\boldsymbol{E}[f(X_\tau, L_\tau, \tau)] = f(0,0,0)$. That is,

$$\boldsymbol{E}[e^{-\mu C}(\gamma \cosh(\gamma C) + (\alpha + \mu)\sinh(\gamma C))e^{-\alpha L(\tau) - \beta\tau}] = \gamma.$$

Note that the argument in the expectation above is always positive for $\gamma \neq 0$ since $(|\mu|/\gamma)\tanh(\gamma C) < 1$ in this case. Thus for $\gamma \neq 0$, formula (7) is verified. If $\gamma = 0$, then $\mu = 0, \beta = 0$, and the formula in the first line of (7) can be obtained by letting γ (hence β) tend to 0 in the second line of (7).

Remark 1. For the case $\alpha > 0, \mu \geqq 0$, formula (7) can alternatively be obtained from a formula given in ([7], Exercise (VI.4.9), part 4°) for $\mu = 0$, by using a Girsanov transformation and the fact that $(\hat{Z}, \hat{L}) \stackrel{d}{=} (|X|, L^0)$ when $\mu = 0$, where $\hat{Z}(t) = X(t) + \hat{L}(t)$, $\hat{L}(t) = -\min_{0 \leqq s \leqq t} X(s)$, and L^0 is the local time at the origin of the unreflected Brownian motion X. However, for $\mu < 0$ this cannot be done straightforwardly due to a problem of integrability. In any case, the formal answer given by the Girsanov transformation was used to guess the candidate function f of (8) which was used for the rigorous proof above.

Remark 2. On setting $\beta = 0$ in (7), we see that $L(\tau)$ is exponentially distributed with mean $(1 - e^{-2\mu C})/2\mu$ if $\mu \neq 0$ or C if $\mu = 0$. The exponential distribution of $L(\tau)$ can also be seen as follows (this argument was told to the author by Jim Pitman). For any $s \geqq 0$, if $L(\tau) > s$, then for $\tau_s \equiv \inf\{t \geqq 0 : L_t > s\}$, $L(\tau) = s + L(\tau \circ \theta_{\tau_s})$ where $\theta_\cdot$ is the shift operator on paths: $\theta_t Z = Z(t + \cdot)$. Since $Z(\tau_s) = 0$, it then follows by the strong Markov property of Z that

$$\boldsymbol{P}(L(\tau) > t + s \mid L(\tau) > s) = \boldsymbol{P}(L(\tau) > t) \qquad \text{for all } t \geqq 0.$$

Thus, $L(\tau)$ is an exponential random variable. To verify the mean of $L(\tau)$, observe that

$$\boldsymbol{E}[Z(t \wedge \tau)] = \mu \boldsymbol{E}[t \wedge \tau] + \boldsymbol{E}[L(t \wedge \tau)],$$

and so on letting $t \to \infty$ we obtain

$$\boldsymbol{E}[L(\tau)] = \boldsymbol{E}[Z(\tau)] - \mu \boldsymbol{E}[\tau] = C - \mu \boldsymbol{E}[\tau].$$

Thus, when $\mu = 0$, we obtain $\boldsymbol{E}[L(\tau)] = C$, and when $\mu \neq 0$, we can deduce the value of $\boldsymbol{E}[L(\tau)]$ provided we have the value of $\boldsymbol{E}[\tau]$ as given in Theorem 2 below. One can check that the formula for $\boldsymbol{E}[L(\tau)]$ obtained in this way agrees with that given above. Alternative derivations of the first line of (7) can be obtained from [7], Exercises (VI.2.10) and (VI.4.12).

By the strong Markov property of Z, τ' is independent of $(L(\tau), \tau)$ and so (5) may be rewritten:

$$\Gamma = \left(\boldsymbol{E}[(L(\tau))^2] - \frac{2(\boldsymbol{E}[\tau L(\tau)] + \boldsymbol{E}[\tau'] \boldsymbol{E}[L(\tau)]) \boldsymbol{E}[L(\tau)]}{\boldsymbol{E}[\tau] + \boldsymbol{E}[\tau']} + \frac{(\boldsymbol{E}[\tau^2] + 2\boldsymbol{E}[\tau]\boldsymbol{E}[\tau'] + \boldsymbol{E}[(\tau')^2])(\boldsymbol{E}[L(\tau)])^2}{(\boldsymbol{E}[\tau] + \boldsymbol{E}[\tau'])^2} \right) \Big/ \boldsymbol{E}[L(\tau)]. \tag{13}$$

By differentiating (7) suitably and letting $\alpha, \beta \to 0$, we can find all moments in the above involving τ and $L(\tau)$. Note in particular that the existence of finite limits of the differentiated expressions as $\alpha, \beta \to 0$ implies existence of the associated moments (cf. Feller [4], p. 435). Since the algebra of these calculations is quite long, Mathematica (Version 1.2) on a SUN 3/60 was used to perform it. The supplier of this software package is Wolfram Research and a comprehensive reference book is [8]. Now, τ' is equal in law to the τ for a reflected Brownian motion with drift $-\mu$. Thus, the first and second moments of τ' can be obtained by replacing μ by $-\mu$ in the corresponding expressions for τ. The formulas thus obtained for the constituent moments appearing in (13) are displayed in the following theorem.

Theorem 2. *For* $\mu = 0$,

$$\boldsymbol{E}[\tau] = \boldsymbol{E}[\tau'] = C^2$$

$$\boldsymbol{E}[\tau^2] = \boldsymbol{E}[(\tau')^2] = \frac{5C^4}{3}$$

$$\boldsymbol{E}[L(\tau)] = C$$

$$\boldsymbol{E}[(L(\tau))^2] = 2C^2$$

$$\boldsymbol{E}[\tau L(\tau)] = \frac{5C^3}{3}.$$

For $\mu \neq 0$,

$$E[\tau] = \frac{\exp(-2\mu C) - 1 + 2\mu C}{2\mu^2}$$

$$E[\tau^2] = \frac{\exp(-4\mu C) + \exp(-2\mu C) + 6\mu C \exp(-2\mu C) + 2\mu^2 C^2 - 2}{2\mu^4}$$

$$E[\tau'] = \frac{\exp(2\mu C) - 1 - 2\mu C}{2\mu^2}$$

$$E[(\tau')^2] = \frac{\exp(4\mu C) + \exp(2\mu C) - 6\mu C \exp(2\mu C) + 2\mu^2 C^2 - 2}{2\mu^4}$$

$$E[L(\tau)] = \frac{1 - \exp(-2\mu C)}{2\mu}$$

$$E[(L(\tau))^2] = \frac{(1 - \exp(-2\mu C))^2}{2\mu^2}$$

$$E[\tau L(\tau)] = \frac{\exp(-2\mu C) - 3\mu C \exp(-2\mu C) - \exp(-4\mu C) + \mu C}{2\mu^3}.$$

When the above expressions are substituted into the right member of (13), after simplification, one obtains the expressions in the right member of (6). Thus, we obtain the same result (6) as Berger and Whitt ([2], (4.14)).

Remark 1. The above expressions for $E[\tau]$ and $E[\tau^2]$ when $\mu = -1$, agree with those that can be obtained from the expressions for $E[T_{0C}]$, $\mathrm{Var}(T_{0C})$ given in Corollary 3.4.1 of Abate and Whitt [1] where $T_{0C} = \tau$.

Remark 2. Note that if one replaces θ by $-\theta$ in (6) and one simplifies by multiplying the resulting numerator and denominator by $\exp(2\theta C)$, then one again has formula (6). Hence, Γ is the same for μ as for $-\mu$. This confirms the fact that one gets the same formula for Γ if $L(\tau)$ in (5) is replaced by $U(\tau + \tau')$ (cf. (4.14) of [2]).

Acknowledgements

The author would like to thank Ward Whitt for bringing this problem to her attention and for helpful comments on a preliminary draft of this paper. Jim Pitman is also thanked for his simple argument that $L(\tau)$ is exponentially distributed. Finally, the author thanks the referee for several helpful comments.

References

[1] ABATE, J. AND WHITT, W. (1988) Transient behavior of the *M/M/*1 queue via Laplace transforms. *Adv. Appl. Prob.* **20**, 145–178.

[2] BERGER, A. W. AND WHITT, W. (1992) The Brownian approximation for rate-control throttles and the *G/G/*1*/C* queue. *Dynamic Discrete Event Systems: Theory and Applications* **2**, 7–60.

[3] CHUNG, K. L. AND WILLIAMS, R. J. (1990) *Introduction to Stochastic Integrations*, 2nd edn. Birkhäuser, Boston.

[4] FELLER, W. (1971) *An Introduction to Probability Theory and Its Applications*, Volume 2. Wiley, New York.

[5] HARRISON, J. M. (1985) *Brownian Motion and Stochastic Flow Systems*. Wiley, New York.

[6] LIONS, P. L. AND SZNITMAN, A. S. (1984) Stochastic differential equations with reflecting boundary conditions. *Comm. Pure Appl. Math.* **37**, 511–537.

[7] REVUZ, D. AND YOR, M. (1991) *Continuous Martingales and Brownian Motion*. Springer-Verlag, New York.

[8] WOLFRAM, S. (1991) *Mathematica — A System for Doing Mathematics by Computer*, 2nd edn. Addison-Wesley, Reading, MA.